Numerical solutions of the singular vortex problem

Cite as: Phys. Fluids 31, 066602 (2019); https://doi.org/10.1063/1.5099896
Submitted: 12 April 2019 . Accepted: 29 May 2019 . Published Online: 24 June 2019

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Submitted: 12 April 2019 • Accepted: 29 May 2019 •
Published Online: 24 June 2019

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ABSTRACT
This study develops a finite-difference numerical formulation to describe the motion of a singular monopole in a quasigeostrophic β-channel model with scale-selective frictional damping, using parameter values typical for the middle-latitude atmosphere and a wide range of viscosities. In this model, the "theoretical" singular vortex is replaced by the equivalent nonsingular vortex of a finite amplitude, consistent with the finite spatial resolution of the model. Numerical experiments demonstrate that at initial stages of the singular-vortex (SV) evolution, this model accurately reproduces the behavior expected from the theoretical considerations of the inviscid case. The numerical model also approximately conserves several invariants of motion derived from the continuous equations and accurately represents their modifications in the presence of friction. The evolution of a singular cyclone in the Northern Hemisphere starts with the development of the dipolar β gyres in the regular component of the flow; these gyres induce initial northward displacement and subsequent westward bending of the cyclone trajectory. At larger times, the β gyres gradually disintegrate, and the singular cyclone in the Northern Hemisphere continues to move northwestward by forming a dipolelike system with the localized secondary regular-field anticyclone northeast of the singular-cyclone center resulting eventually in a friction-assisted steady-state regime. The SV trajectories tend to become more zonally elongated for large vortices and small values of viscosity. These results lay a foundation for numerical consideration of systems of multiple singular vortices, which could provide further insights into our fundamental understanding of the processes underlying the multiscale atmospheric and oceanic variability.

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I. INTRODUCTION
One of the most interesting and important properties of the geophysical vortices is their ability to self-propel relative to the background flow. Advection of the ambient fluid by the vortices results in redistribution of the background potential vorticity (PV), which generates an anomalous velocity field on top of that of the background flow and vortices themselves. This additional velocity field interacts with the vortices and affects vortex trajectories, thereby resulting in further modifications of the large-scale flow. Such complex interactions have been actively studied from the late 1980s up until now both in geophysical fluid dynamics (e.g., McDonald, 1999; Reznik and Kizner, 2010; and Koshel et al., 2019) and in plasma physics (e.g., Tur and Yanovsky, 2017).

The above processes play an important role in the general circulation of the atmosphere and the ocean, both of which are characterized by multiscale flows with a vigorous eddy field of intense cyclones and anticyclones (synoptic eddies) that control atmospheric and oceanic weather variations and exhibit a pronounced low-frequency variability. Recent atmospheric studies (Löptien and Ruprecht, 2005; Kravtsov and Gulev, 2013; and Kravtsov et al., 2015) showed that the latter variability can be thought of in terms of the low-frequency redistribution of the atmospheric storm tracks, which is insensitive to the detailed spatial structure of the individual synoptic eddies. This observation suggests an idea of tackling the problem of multiscale midlatitude atmospheric variability by considering the systems of interacting singular vortices (Obukhov, 1949; Morikawa, 1960; Gryanik, 1983a; 1983b; 1986; and Reznik, 1992); in such systems, the singular eddy field is clearly isolated from the regular field, thereby allowing unambiguous identification of the various dynamical components of the eddy–mean-flow interaction. The same
approach is applicable for studying oceanic mesoscale processes (Early et al., 2011).

The equations describing the evolution of an arbitrary number of singular vortices embedded into a background flow have been formulated previously (Reznik, 1990; Reznik; and Kizner, 2007;a; 2007b; and 2010) but are only amenable to analytical treatment in a very limited number of special cases. One of the main goals of this paper is to present a numerical scheme which would accurately capture the dynamics of such “discrete-continuous” systems. To provide proof-of-concept, we focus here on the evolution of a singular monopole on a β plane.

There is an extensive literature on the dynamics of the β-plane monopoles, both distributed and singular, but some questions still remain unanswered. For example, it was shown analytically (e.g., Reznik, 1992; Reznik and Drewar, 1994; and Sutyrin and Fliter, 1994), numerically (e.g., Sutyrin et al., 1994; Carnevale, 2001), and experimentally (e.g., Carnevale et al., 1991) that the self-propelled northward (southward) drift of a cyclone (anticyclone) is induced by the so-called β-gyres—secondary dipolar circulation arising due to Rossby waves radiated by the vortex. This mechanism is definitely dominant at initial, linear stages of the vortex motion, but does it continue to operate later on, when nonlinear effects become important? Our present numerical experiments show, among other things, that, generally, this is not the case, and the long-time evolution of the monopole is governed by different dynamics.

The rest of the presentation is organized as follows. Section II formulates the general problem and provides some theoretical background. In Sec. III, we develop a numerical finite-difference model describing the evolution of a singular monopole on a β plane and further show, in Sec. IV, that it provides a reasonable approximation of the continuous equations by comparing its numerical and (approximate) analytical solutions and considering the invariants of motion. Section V analyzes the dependence of the singular vortex evolution on the size of the vortex and explores the effects of friction. These results are summarized and further discussed in Sec. VI.

II. THEORETICAL BACKGROUND

A. General formulation

The conservation of quasigeostrophic (QG) potential vorticity (PV) in a 1-layer or 1½-layer (x, y) β-plane model states,

\[ \partial_t (\tilde{\Omega} + \beta y) + J(\tilde{\psi}, \tilde{\Omega} + \beta y) = 0, \]  

(1a)

\[ \tilde{\Omega} = \nabla^2 \tilde{\psi} - a^2 \tilde{\psi}, \]  

(1b)

where \( \tilde{\psi} \) is the streamfunction, \( \tilde{\Omega} \) is the relative vorticity, \( \tilde{\Omega} = \nabla^2 \psi - a^2 \psi \) is the PV, \( a = R \Omega \), and \( R \) is the Rossby scale; in the above equation, the partial derivative with respect to a variable \( z \) is denoted as \( \partial_z \), \( \nabla^2 = \partial_x^2 + \partial_y^2 \) is the 2D Laplacian, \( t \) is time, and \( J(f, g) = \partial_x f \partial_y g - \partial_y f \partial_x g \) is the Jacobian. Consider a discrete–continuous system consisting of singular vortices embedded into the regular flow and decompose the total flow as follows (Reznik, 1992):

\[ \psi = \psi_0 + \psi_s; \quad \tilde{\Omega} = \Omega + \Omega_s; \quad \Omega = \nabla^2 \psi - a^2 \psi; \quad \Omega_s = \nabla^2 \psi_s - a^2 \psi_s, \]  

(2)

where the variables without the subscript denote the regular flow component and those with the subscript \( s \) denote the singular flow component. The latter component is represented by the sum of \( N \) Bessel vortices,

\[ \psi_s = \sum_{n=1}^{N} \psi_{sn}, \]  

(3a)

\[ \psi_{sn} = \frac{A_n}{2\pi} \frac{1}{\sqrt{\log (r - r_n(t))}}. \]  

(3b)

Here, the singular-vortex (SV) amplitudes \( A_n \) and horizontal scales \( p_n^{-1} \) are prescribed and constant, and the vortex trajectories \( r_n(t) = (x_n(t), y_n(t)) \) are determined in the process of the solution. The singular flow’s relative vorticity is given by

\[ \Omega_s = \sum_{n=1}^{N} A_n \delta(x - x_n(t)) \delta(y - y_n(t)) + (p_n^2 - a^2) \psi_{sn}, \]  

(4)

where \( \delta \) is the Dirac delta function.

Defining the singular vortex \( x \)- and \( y \)-velocity components as \( U_n = \dot{x}_n \) and \( V_n = \dot{y}_n \) (where the dot denotes the time derivative) and substituting (2) into (1) give, in view of (3) and (4),

\[ \partial_t \Omega + J(\psi + \psi_s, \Omega + \beta y) + \sum_{n=1}^{N} (p_n^2 - a^2) \left( \psi + \psi_{sn} - u_n x - V_n y \right) = 0; \]  

(5)

with

\[ U_n = -\partial_y (\psi + \psi_{sn})_{r=r_n}, \quad V_n = \partial_x (\psi + \psi_{sn})_{r=r_n}. \]  

(6)

Equation (5) determines the evolution of the regular-flow streamfunction, and Eq. (6), the SV trajectories. If the β effect and regular component are absent (\( \beta = \psi = 0 \)) and \( p_n = a \) [in which case, the singular vortices all become the point vortices as per (4)], then (5) is satisfied identically and (6) is reduced to the well-known Hamiltonian system of ordinary differential equations for coordinates of interacting SV (cf. Saffman, 1992). However, if \( \beta \neq 0 \) and/or the scale of at least one of the vortices (assuming \( N > 1 \)) differs from \( R \), then the regular component \( \psi \) is necessarily generated, even if it is absent at initial time. To describe the motion in this general case, one should solve the full coupled “discrete-continuous” system (5) and (6).

B. Individual singular vortex

We now consider a single vortex \( (N = 1) \), in which case the singular component of streamfunction (3) reduces to

\[ \psi_s = -\frac{A}{2\pi} \frac{1}{\log \left(1 - p_n^{-1} \right)}, \]  

(7)

and (5) and (6) become

\[ \partial_t \Omega + J(\psi + \psi_s, \Omega + \beta y) + \left( p_n^2 - a^2 \right) \left( \psi + U_0 y - V_0 x, \psi_s \right) = -K \nabla^2 \psi \]  

(8)

and

\[ U_0 = \dot{x}_0 = -\partial_y \psi_{s0}, \quad V_0 = \dot{y}_0 = \partial_x \psi_{s0}. \]  

(9)
respectively. The initial regular field is assumed to be zero, i.e.,
\[ \psi|_{t=0} = 0. \] (10)

Note that anticipating the development of a numerical formulation for the system (7)–(9) in Sec. III, we added the horizontal superviscosity with (positive) coefficient \( K \) in the right-hand side of (8); this scale-dependent frictional damping acts on the regular field only (and not on the singular vortex, which maintains constant shape), efficiently suppressing small-scale features and providing stability of the numerical scheme. Hence, in numerical modeling of inviscid processes, the value of \( K \) is minimized given the model’s (sufficiently high) spatial resolution. Indeed, in our quasi-inviscid experiments presented in Sec. IV (which will hereafter be referred to as standard experiments), we demonstrate that stable simulations of (7)–(9) with high spatial resolution and very small \( K \) approximate well the behavior of the continuous inviscid system.

The scale-selective frictional damping of the regular flow field generated by the singular vortex, while ensuring the model’s numerical stability even for small \( K \), can also be interpreted to parameterize the effects of subgrid-scale processes on the regular flow. Therefore, besides the standard near-inviscid experiments with minimum viscosity and maximum spatial resolution, we also study a wide range of model resolutions and superviscosities to analyze the potential effects of frictional parameterization on the dynamics of the vortex (see Sec. V). These effects can play an important role, for example, in the behavior of synoptic storms simulated by numerical weather prediction models.

A key assumption in our formulation of the viscous effects is that the singular vortex is not influenced by the friction. This assumption is of little importance for interpretation of the standard quasi-inviscid experiments, because in this case, the minimal superviscosity included in the equations of motion is but a means to provide the model’s numerical stability. However, in experiments with enhanced superviscosity, it is equivalent to assuming that the constant-shape singular vortex (7) is maintained by dynamical balances unresolved by our model (e.g., baroclinic instability or thermodynamic processes). Wu and Emanuel (1993) used a similar approach in their idealized simulations of tropical cyclones, which are maintained by the diabatic processes not explicitly represented in the dry, quasigeostrophic system.

1. Energy conservation

The energy conservation takes the form
\[ \partial_t E_r - (p^2 - a^2)E_{\nu} - A \psi|_{r=r_0(t)} = D_1 + D_2, \] (12a)
where
\[ E_r = \frac{1}{2} \int_R ((\nabla \psi)^2 + a^2 \psi^2) dxdy \] (12b)
is the energy of the regular flow, the term
\[ E_{\nu} = \int_R \left( \psi \psi_r + \frac{\psi_r^2}{2} \right) dxdy \] (12c)
describes the energy supply associated with the singular vortex [note that the second term in (12c) does not depend on time and its time derivative in (12a) is, therefore, zero], and the terms
\[ D_1 = K \int_R \psi \nabla^4 \psi dxdy; \quad D_2 = K \int_R \psi_r \nabla^6 \psi dxdy \] (12d)
are due to the presence of frictional damping in model (8). Note that while the term \( D_1 \) is negative definite
\[ D_1 = -K \int_R \left( \nabla (\nabla^2 \psi) \right)^2 dxdy, \] (12e)
the term \( D_2 \) is not necessarily so as
\[ D_2 = K \left( A \nabla^4 \psi|_{r=r_0(t)} + A p^2 \nabla^2 \psi|_{r=r_0(t)} + A p^4 \psi|_{r=r_0(t)} + p^4 \psi_1^2 \right). \] (12f)
(Here and below, the overline denotes the spatial integral over \( R \).)

2. Enstrophy integral

The enstrophy integral in fact involves a combination of the energy- and enstrophy-related terms. The corresponding conservation law can be written as follows:
\[ \partial_t (S - \beta A y_0) = D_3, \] (13a)
where
\[ S = \frac{1}{2} \int_R ((\nabla^2 \psi)^2 + (\nabla^2 a^2) (\nabla \psi)^2 + p^2 a^2 \psi^2) dxdy \] (13b)
and
\[ D_3 = -K \int_R \left( \nabla^4 \psi \right)^2 dxdy + p^2 D_1. \] (13c)
Since both terms comprising \( D_3 \) are negative definite, the quantity \( S - \beta A y_0 \) in (13a) can only decrease with time.

3. Momentum integral

Multiplying (8) by \( y \) and integrating over \( R \) result in the expression for the conservation of the \( y \) component of momentum, \[ \partial_t (\overline{y \psi} + y_0(t) \overline{\psi}) = 0. \] (14a)
Here, \( \overline{\psi} \) can be computed analytically using (11) with \( F = 1 \),
\[ \overline{\psi} = -A p^{-2}. \] (14b)
The expression for the conservation of the x component of momentum (not shown) can be obtained in an analogous way by forming the product of (8) and x (see Reznik and Kizner, 2007a).

4. Regular vorticity at the center of the singular vortex

According to (9), the singular vortex moves with the fluid and thus marks a fluid parcel. Since the singular streamfunction remains constant in the reference frame associated with the singular vortex, the conservation of potential vorticity (8) at \( r = r_0(t) \) in the absence of friction (\( K = 0 \)) gives the expression

\[
q \equiv (\nabla^2 \psi - a^2 \psi)_{r=r_0(t)} + \beta y_0(t) = \beta y_0(0) = \text{const},
\]

(15)

where we assumed that there is no regular flow at \( t = 0 \). The expression (15) allows one, by knowing the trajectory of the singular vortex, to determine the value of the regular potential vorticity at the point of the singular vortex at any time. This constraint is approximately valid with small nonzero damping in (8) and can thus be used, among other things, to estimate the ability of the numerical scheme developed in Sec. III to reproduce nonviscous dynamics of the system. We will also use (15) to formulate an alternative integration scheme for (8) in the vicinity of the singular vortex (Sec. IV B).

III. NUMERICAL FORMULATION

A. General description

Equations (7)–(9) were discretized on an equally spaced regular grid in an x-periodic channel of length \( L_x \) and width \( L_y \) using second-order accuracy centered differences in space subject to no-flow and no-slip conditions on zonal boundaries (\( \psi_x = \psi_y = \psi_{yy} = 0 \)), the 4th-order Arakawa scheme for advection (Arakawa, 1966), and leapfrog time integration scheme, as well as mass and momentum constraints (McWilliams, 1977) and integrated forward from the state of rest \( \psi = 0 \). Note that the channel width \( L_y \) we chose was large enough that in the course of all experiments, the streamfunction fields and their derivatives remained very close to zero near the zonal boundaries. Furthermore, the duration \( T \) of each simulation was chosen to correspond to the time it takes the fastest Rossby wave to travel the channel x period: \( T \approx L_x/\beta R_0^2 \). As we will see, the zonal propagation velocity of the SV does not exceed the limiting Rossby wave speed \( \beta R_0^2 \), so restricting \( T \) in this way ensures that the Rossby-wave trail left behind by the singular vortex does not interfere with the vortex in the course of the simulation. Both of these conditions make the present simulations completely analogous to the case of an unbounded domain despite the model’s channel geometry.

The Bessel vortex streamfunction \( \psi_s \) in (7) has a singularity at the center of the vortex, which presents an obvious difficulty in integrating the numerical representation of (7)–(9) with a finite spatial resolution \( \Delta x = \Delta y = \Delta \). The streamfunction \( \psi_s \) is the solution of the Helmholtz problem,

\[
\nabla^2 \psi_s - \beta^2 \psi_s = A \delta(x - x_0) \delta(y - y_0).
\]

(16)

To overcome the above difficulty, we replaced the analytical Bessel vortex \( \psi_s \) with its finite-difference replica \( \psi_s^* \) obtained by numerically inverting the finite-difference equation analogous to (16),

\[
(\psi_s^*_{i+1,j} + \psi_s^*_{i-1,j} + \psi_s^*_{i,j+1} + \psi_s^*_{i,j-1} - (4 + \beta^2 \Delta^2) \psi_s^*_{i,j})/\Delta^2 = A \delta_{i\omega} \delta_{j\theta}/\Delta^2,
\]

(17)

where \( \psi_s^* \) is the singular streamfunction at the grid point \((i,j)\), the grid point \((\hat{\theta}, \hat{\rho})\) is located at the center of the singular vortex, and \( \delta_{i\omega} \) is the Kronecker delta, which is equal to 1 if \( i = j \) and is zero otherwise. Note that the sum \( \sum_{i,j} \Delta x_\omega \Delta y_\theta A \delta_{i\omega} \delta_{j\theta}/\Delta^2 \) approximating the area integral of the right-hand side of (17) is also equal to \( A \), thereby confirming that it tends to the Dirac delta function as the model resolution \( \Delta \to 0 \).

The numerical solution of (17) for the case \( \Delta = 25 \) km, \( R_0 = 600 \text{ km} \), and \( p^{-1} = 600 \text{ km} \) is shown in Fig. 1, with the vortex centered at a grid point in the middle of the channel in \( x \) and \( y \) of the channel width in \( y \) placement was used as the initial condition in all of our numerical experiments. The streamfunction in Fig. 1 was nondimensionalized using the velocity scale of \( U = 10 \text{ ms}^{-1} \) and the spatial scale \( \Delta \), with the corresponding streamfunction scale \( \Delta U \). This numerical solution coincides with the analytical profile (7) at all grid points except for the center of the vortex, where, in contrast to the analytical expression, the numerical solution is finite, consistent with the finite resolution of the model grid. Another possible approach to avoid the singularities in (7)–(9) will be discussed briefly in Sec. IV B.

For an arbitrary time \( t \), the singular streamfunction \( \psi_s \) was computed by shifting the 2D finite-difference profile \( \psi_s^* \) to the current vortex location \( r_0(t) \) (which does not have to belong to the model grid) and interpolating onto the model grid in the vicinity of the singular vortex (not shown).
(±50 grid points) of the vortex center using cubic splines; outside of this region, \( \psi \) was computed directly via (7). Cubic splines were also used to compute the vortex velocity \( U_0(t), V_0(t) \) via spatial derivatives of the regular streamfunction \( \psi \) [Eq. (9)].

Hence, knowing the singular vortex location \( \psi(t) \) and regular streamfunction \( \psi \) at time \( t \), we can compute the singular streamfunction field \( \psi \), on the model grid, as well as the singular vortex velocity \( U_0(t), V_0(t) \), which allows us to find the regular potential vorticity \( \nabla^2 \psi - \alpha^2 \psi \) and, after inversion, the regular streamfunction \( \psi \) at the next time step from (8), along with the new singular vortex location from (9).

### B. Model parameters and simulations

All model parameters are listed in Tables I and II. In our experiments, we chose the “environmental” parameter values typical for the midlatitude atmosphere (Table I) and varied two control parameters: the size of the singular vortex \( L_0 = p^{-1} \) (Table I) and the superviscosity \( K \), which also required adjusting the model spatial resolution and time step (Table II). We used three different values for \( L_0 \) (“small vortex,” “point vortex,” and “large vortex” cases, respectively) and four different values of \( K \) (hereafter, “large friction,” “medium friction,” “small friction,” and “tiny friction” cases), with the \( K \) values differing by three orders of magnitude between the large-friction and tiny-friction cases. Note that the range of spatial resolutions \( \Delta = 200, 100, 50, \) and 25 km (coarse, intermediate, high, and very high resolutions) corresponding to the decreasing \( K \) covers the values typical of the global climate models on one end to those of mesoscale-resolving regional numerical weather prediction models on the other, which makes our sensitivity analysis particularly relevant for understanding possible perturbations of the storm-track dynamics in coarser-resolution climate-type models.

In addition to the experiments reflecting 12 combinations of the control parameters introduced above, we performed an experiment tagged as the “convergence check” in Table II, which combined high resolution and medium friction to check the convergence of the solution in the intermediate resolution/medium friction case.

### TABLE I. Resolution-independent model parameters.

<table>
<thead>
<tr>
<th>Parameter notation/value</th>
<th>Parameter description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_x = 51 \text{ 200 km} )</td>
<td>x extent of the channel</td>
</tr>
<tr>
<td>( L_y = 30 \text{ 000 km} )</td>
<td>y extent of the channel</td>
</tr>
<tr>
<td>( T = 80 \text{ days} )</td>
<td>Duration of each simulation</td>
</tr>
<tr>
<td>( \beta = 2 \times 10^{-11} \text{ m}^{-1} \text{s}^{-1} )</td>
<td>y gradient of the Coriolis parameter</td>
</tr>
<tr>
<td>( R_d = 600 \text{ km} )</td>
<td>Rossby radius of deformation</td>
</tr>
<tr>
<td>( A = 2\pi \times 5\beta R_d^2 )</td>
<td>Amplitude of the vortex (intense vortex case; cf. Reznik, 1990; 1992)</td>
</tr>
<tr>
<td>( 300 \text{ km} )</td>
<td>Vortex size: small-vortex case</td>
</tr>
<tr>
<td>( 600 \text{ km} )</td>
<td>Vortex size: point-vortex case</td>
</tr>
<tr>
<td>( 1200 \text{ km} )</td>
<td>Vortex size: large-vortex case</td>
</tr>
<tr>
<td>( a = R_d^{-1} )</td>
<td>Inverse Rossby radius</td>
</tr>
<tr>
<td>( p = L_v^{-1} )</td>
<td>Inverse singular vortex size</td>
</tr>
</tbody>
</table>

Yet another sensitivity experiment used an alternative, semiteoretical scheme for determination of the regular vorticity in the vicinity of the singular vortex (Sec. IV B). Both of these additional experiments produced the results nearly identical to those in the corresponding control experiments, thereby confirming the validity of our numerical model.

### IV. RESULTS

The following presentation is organized as follows. We start with the discussion of the results pertaining to the highest-resolution tiny-friction parameter regime, which are expected to best correspond, at the initial stages of the evolution, to the inviscid theory of the singular vortices presented briefly in the Appendix (see also Reznik, 1990; 1992). In this discussion, we aim to demonstrate that our numerical model provides a faithful approximation of the underlying continuous equations (as listed in Sec. II B) and approximately conserves the integrals of motion (derived in Sec. II C). We describe the joint evolution of the singular vortex and the regular flow throughout the simulation period and estimate nonlinear effects at the later stages of the system development, beyond the linear regime predicted by the analytical considerations in the Appendix. We next turn to the effects of the scale-selective frictional damping; these effects become particularly important at large times in the simulations corresponding to the parameter regimes similar to those in coarser-resolution climate models and are thus bound to impact the simulated storm-track dynamics in such models.

### A. Salient features of the quasi-inviscid singular vortex evolution

#### 1. Initial evolution

The evolution of the flow in the tiny-friction point-vortex case is summarized in Figs. 2 and 3; the evolution of small or large vortices is qualitatively similar (not shown here, to be discussed further in Sec. V). The motion of an intense monopole on the \( \beta \) plane is due to its interaction with the regular flow, which itself represents Rossby waves radiated by the vortex (Reznik, 1992; 2010; and McDonald, 1999). In particular, the cyclone (\( A > 0 \), counterclockwise rotation in the Northern Hemisphere) advects fluid parcels west of its center to the south and those east of the center to the north, which generates positive (cyclic; \( \psi < 0 \)) vorticity anomalies to the west and negative (anticyclonic; \( \psi > 0 \)) vorticity anomalies to the east of the vortices.

Yet another sensitivity experiment used an alternative, semiteoretical scheme for determination of the regular vorticity in the vicinity of the singular vortex (Sec. IV B). Both of these additional experiments produced the results nearly identical to those in the corresponding control experiments, thereby confirming the validity of our numerical model.

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FIG. 2. Evolution of the regular streamfunction $\psi$ for the point-vortex ($L_v = R_d = 600$ km) 25-km-resolution (tiny friction) simulation (see Tables I and II). The streamfunction snapshots are shown in the $25,000 \times 10,000$-km region centered at the current location of the singular vortex, marked by the black dot in each panel (see Fig. 3(a) for the vortex trajectory). Black contours show zero isoline of the streamfunction. Time is given in panel captions.

cyclone due to potential vorticity conservation. By the same process, these secondary cyclones and anticyclones produce their own satellite cyclones and anticyclones, with the parent cyclone generating a daughter cyclone to the west and a daughter anticyclone to the east of itself, and the parent anticyclone generating a daughter anticyclone to its west and a daughter cyclone to its east.

The secondary cyclone and anticyclone in the immediate vicinity of the main vortex constitute the $\beta$-gyre dipole, which is clearly seen in Fig. 2 for times up to $\approx 30$ days. The $\beta$ gyres advect the vortex along the dipole axis according to (9); therefore, the very initial motion of the singular cyclone due to its interaction with the west-to-east oriented $\beta$-gyre dipole is northward (Fig. 3). However, the intense cyclone quickly turns the dipole axis counterclockwise (Fig. 2, $t = 1$ day), which generates westward component of the motion, and the whole system continues to propagate westward and to the north (Fig. 3), leaving behind a trail of alternating cyclonic/anticyclonic secondary anomalies (Fig. 2), consistent with the above qualitative discussion of Rossby-wave generation by the singular vortex.

Approximate theory of the $\beta$ gyres is presented in the Appendix. In this theory, Eq. (A2) is replaced by the approximate linear equation (written in the reference frame associated with the vortex),

$$\partial_t (\Omega + \beta y) + \mathbf{J}(\psi_s, \Omega + \beta y) + \left( p^2 - a^2 \right) \mathbf{J}(\psi + U_0 y - V_0 x, \psi_s) = 0,$$

with conditions (9) and (10). For the point vortex case, $p = a$, and Eq. (18) can be readily solved with the solution being given by formulas (A6) and (A7). This solution is exactly reproduced by the corresponding numerical simulation of (18) (see Fig. 3). For the vortices with $p \neq a$, the solution to the problems (18), (9), and (10) is found numerically. The solution (A6) and (A7) in this case turns out to be valid only at small times for which $\psi$ remains sufficiently small (approximately equal to $t \sim 1$ day, per the results of numerical experiments). At longer times, the last term

FIG. 3. Singular vortex path and velocity components for the point-vortex ($L_v = R_d = 600$ km) 25-km-resolution (tiny friction) simulation (see Tables I and II). (a) Vortex trajectories, sampled at three-day intervals: red circles—numerical simulation; blue x-symbols—the theoretical solution of the inviscid ($K = 0$) point-vortex problem, formally valid at small times only (Reznik, 1990; 1992; the Appendix here); and green plus signs—numerical simulation; blue x-symbols—the theoretical solution of the inviscid ($K = 0$) point-vortex problem, formally valid at small times only (Reznik, 1990; 1992; the Appendix here); and green plus signs—numerical solution of the linearized problem (the Appendix). (b) and (c) Time series of the $x$ and $y$ components of the singular-vortex velocity: red dashed line—numerical simulation; black line—theoretical solution; and green dashed line—solution of the linearized problem. The coordinates $x$ and $y$ are given in km, velocities in ms$^{-1}$, and time in days.
in the left-hand side of (18) becomes of importance and degrades the match between the analytical and numerical trajectories (not shown).

For all vortex sizes, however, the vortex trajectories from the full equations (8) (note that these equations are written in the absolute reference frame) and linearized equations (18) (written, once again, in the reference frame attached to the vortex) are close to each other up to a time $t_{\text{fin}}$ on the order of 20–30 days (see Figs. 3 and 10). The time $t_{\text{fin}}$ thus exceeds much the typical time scale associated with vortex rotation, $t_{\text{v}} = 2\pi^{-1}/A \approx 0.2–0.4$ days. Scaling estimates of the terms $\beta \psi_x + J(\psi + U_0 y - V_0 x, \Omega)$ in Eq. (A2) neglected in the linear solution show that these terms should cease to be small after the period $t_0 \sim 1/\beta R_0^2$. In all cases, however, the scaling estimate $t_0$ is, again, very small compared to $t_{\text{fin}}$, e.g., for the point vortex $t_0 = 1/\beta R_0^2 \approx 1$ day and $t_{\text{fin}} \gg t_0$. Such long-term applicability of the solution of the linearized singular-vortex equations arises in large part due to a substantial compensation between the terms $\beta \psi_x$ and $J(\psi + U_0 y - V_0 x, \Omega)$ so that their sum remains smaller than the individual terms until the regular field $\psi$ becomes less localized and expands beyond the immediate vortex vicinity. We note that the applicability of the linear theory of $\beta$ gyres at times much longer than expected from scaling arguments was documented in a large number of studies on distributed vortices (e.g., Reznik and Dewar, 1994; Sutyrin et al., 1994; Reznik et al., 2006; and Sutyrin and Morel, 1997).

2. Change of self-propelling mechanism at later stages

At larger times, two interesting things happen with regard to further interaction between the singular cyclone and the $\beta$-gyre dipole. First, while initially the center of the singular cyclone is naturally located right on the dipole’s axis (Fig. 2, $t = 1$ day), at later times, it gets sucked in the anticyclonic ($\psi < 0$) lobe of the dipole and stays there throughout the course of the simulation, while remaining to be slightly offset from the center of this lobe, leading to the north-westward propagation of the “coupled” singular-cyclone/regular-anticyclone flow system. Second, starting from about $t \approx 22–29$ days, when the center of the singular vortex gets fully embedded in the positive lobe of the $\beta$ gyre dipole, the negative lobe of this dipole gets shielded from the singular vortex and eventually dissipates. Thus, at large times, the $\beta$ gyres disintegrate and the singular cyclone in the Northern Hemisphere continues to move north-westward by forming a dipole-like system with the localized secondary regular-field anticyclone northeast of the singular-cyclone center.

3. Considerations using integrals of motion

Evolution of the integrals of motion derived in Sec. II C is shown in Fig. 4 in all of the tiny-friction simulations (for small-vortex, point-vortex, and large-vortex cases). The essential message from Fig. 4 is that the changes in all of the invariants considered due to the presence of friction remain small compared to the changes in the individual terms comprising any given invariant. This conservation is best in the large-vortex simulation since the $\beta$-gyre dipole. The term in (12a) proportional to $E_0$ [see (12c)] is zero for the point-vortex case $p = a$, so for small frictional damping,

$$\frac{\partial \psi}{\partial t} \bigg|_{\text{linear}} = \frac{1}{\Omega} \frac{\partial E_r}{\partial t} > 0,$$

since the energy of the regular flow $E_r$ also monotonically increases, as seen from Fig. 4. Therefore, $\psi_{\text{linear}}(t)$ increases with time, meaning that the center of the singular vortex migrates from the $\beta$-gyre dipole axis ($\psi = 0$) toward and into its positive lobe ($\psi > 0$). The same argument is valid for the small and large vortices because in both cases, the sum $E_r = (\bar{p}^2 - \bar{a}^2)E_0$ increases with time (see Fig. 4).

4. Regimes of singular-vortex evolution

The singular-cyclone trajectory thus exhibits several regimes. We will call the initial stage of flow development the linear regime since during this stage, the last two terms in the left-hand side of (A2) are small and the numerical solution follows the linear problem (18), (9), and (10), with vortex trajectory starting northward and then quickly bending toward west (Fig. 3). The solution of the linearized problem is formally valid up to time $t_0$ (see above) but, in reality, stays fairly close to the full numerical solution much longer.

In due course, the regular field grows and expands beyond the immediate vicinity of the singular vortex so that the terms neglected in (18)—most importantly, the nonlinear term $J(\psi, \Omega)$ in (A2) describing self-interaction of the regular field—become significant. As this takes place, the system enters the inertial regime, in which the simulated trajectory deviates from the trajectory of the solution of the linearized problem, resulting, first, in a relative slowdown of the zonal (westward) vortex velocity $U_0$ combined, later on (at times longer than $\sim t > 20$ days), with the eventual speed up of the meridional (northward) vortex velocity $V_0$, leading to the northward deflection of the simulated vortex trajectories from the trajectory based on the solution of the linearized problem (Figs. 3 and 10). Similar behavior is observed in the evolution of distributed vortices (see, for example, Reznik et al., 2000, Fig. 10; Sutyrin et al., 1994, Fig. 4). The change in the self-propelling mechanism from the $\beta$ gyres to the embedded vortex layout occurs during the inertial stage of vortex evolution described above. This stage is also characterized by a weak long-period ($\sim 40$ days) oscillation of the singular vortex velocities around their mean values (Figs. 3 and 10). An additional experiment in the channel twice the width of the original channel produces an identical behavior (figure not shown), which demonstrates that this oscillation is not associated with the
FIG. 4. Integrals of motion in the 25-km-resolution (tiny friction) simulations (see Table I). The left, middle, and right columns correspond to the small-vortex, point-vortex, and large-vortex cases, respectively (see Table I). Top row—the potential vorticity \( q(t) \) at the location of the singular vortex (blue curves) [Eq. (15)], along with the time series of \( \beta_y y_0(t) \) (red curves). The two curves in each panel are intentionally offset by using different values of \( y_0(0) \). Second row—conservation of the \( y \) component of momentum [Eq. (14)]. Red curves: \( \psi/\psi_s \); blue curves: \( y_0(t) \); and black dashed curves: \( \psi/\psi_s + y_0(t) \). All terms are normalized by the width of the channel. Third row—energy conservation [Eq. (12)]. Red curves: \( E_r \); magenta curves: \( - (p^2 - a^2) E_r \); blue curves: \( - A \psi/\psi_s \); black curves: \( E_r - (p^2 - a^2) E_r - A \psi/\psi_s \); and green dashed curves show the time integral of the right-hand side \( D_1 \). All terms are normalized by the area of the channel; units are m\(^2\) s\(^{-2}\). Bottom row—enstrophy integral [Eq. (13)]. Red curves: \( S \); blue curves: \( - \beta Ay \); black curves: \( S - \beta Ay \); and green dashed curves show the time integral of the right-hand side \( D_2 \). All terms are dimensionless. Time is in days.

B. Effects of scale-selective damping and alternative numerical scheme

1. Potential vorticity (PV) homogenization

Despite the little-to-no apparent effect on the invariants of motion, the scale-selective frictional damping in (8) does play a role in the system’s dynamics even in the tiny-friction simulations. To illustrate these effects, we consider the tiny-friction point-vortex case, which, as shown above, nearly conserves the regular potential vorticity at the center of the vortex throughout the course of the simulation (Fig. 4). Figure 5 shows initial stages of the evolution of the regular-flow PV \( \nabla^2 \psi - a^2 \psi + \beta y \) in the reference frame associated with the singular vortex. Advection of PV by the singular vortex results in the spiral PV structure with progressively smaller radial scales toward the center of the vortex (Fig. 5, frame 1); this structure is entirely consistent with the theoretical analysis of the inviscid case by Reznik, 1990; 1992. Due to the presence of scale-dependent frictional damping \(-K \nabla^6 \psi \) in (8), however, these small-scale anomalies are efficiently mixed at later times, resulting in the complete PV homogenization within the core of the vortex (Fig. 5, frames 30–40). The same effect takes place for the large-vortex and small-vortex cases. The fact that the numerical solution for the point vortex is very close to the analytic inviscid solution \((A6) \) and \((A7) \) for times up to 10 days and longer (Sec. IV A) means that the small-scale PV anomalies in the vicinity of the singular-vortex center do not significantly affect the regular flow field. However, this homogenization effect allows suggesting an alternative numerical scheme for the system \((7)-(9) \).
2. Alternative numerical scheme

One of the essential questions in the present study is whether the system (7)–(9) discretized with the use of the finite-difference singular-vortex analog (17) represents a reasonable approximation of the true continuous singular-vortex system. The main difference between these two systems is in the structure of the singular stream-function $\psi_s$ in the vicinity of the singular-vortex center (Fig. 1). The finite-difference approximation to $\psi_s$ (17) does provide bounded singular-vortex velocities and finite advection terms in this region to aid a numerically efficient and stable simulation with a relatively large time step, but could it result in the errors in the numerically simulated evolution compared to what is expected from the theory?

To address this question, we developed an alternative numerical scheme to determine the regular PV in the vicinity of the singular-vortex center by utilizing the PV constraint (15) and the PV homogenizations due to scale-selective frictional damping and applied it to the least-damped large-vortex, tiny friction case. In this scheme, Eqs. (8) and (9) were integrated forward in time as usual, except that, at each time step, the PV at the four grid points closest to the new location of the singular vortex was all updated by the value of PV at the center of the vortex found from the Lagrangian conservation of potential vorticity (15), after which the streamfunction was computed as before by inverting the updated elliptic problem. Obviously, in this case, there is no difference between the “exact” singular vortex (7) and its finite-difference analog since the points of singularities are excluded from calculation of derivatives. The resulting numerical solution (figure not shown) is nearly identical to the original solution thereby providing further evidence that the discretized version of the singular vortex problem is a fair approximation to its continuous formulation. We note that the alternative scheme proposed here can in principle be improved further by accounting for the expansion of the homogenization domain with time seen in Fig. 5.

In summary, we showed that our numerical representation of the singular-vortex system (7)–(9) provides, under high-resolution/tiny-friction parameter regime, an accurate replica of the continuous equations and approximates well the inviscid case.

V. SENSITIVITY TO PARAMETERS AND EFFECTS OF FRICTION

The qualitative evolution of the singular/regular flow system for different values of $K$ and $L_v$ is analogous to that in the tiny-friction case described in Sec. IV A. For example, the regular streamfunction in the point-vortex large-friction case (Fig. 6)
exhibits the same initial development of β gyres, generation of the Rossby-wave trail, and, later, merger of the singular cyclone with the anticyclonic lobe of the β gyres, as well as the separation and dissipation of the cyclonic lobe of the β gyres (compare Figs. 2 and 6). Since all these features appear across the entire range of the K and Lv parameters considered, including a quasi-inviscid tiny-friction case, we conclude that they are dynamically independent of the frictional processes. However, the frictional effects do become quantitatively important in the long-term evolution of the system considered at large values of the superviscosity K (see Sec. V C).

A. Invariants of motion

To further validate our numerical model and gauge the extent to which frictional dissipation affects its solutions, we first computed the budgets associated with various terms of the singular vortex system’s conservation integrals (Sec. II C). The friction enters the energy and enstrophy integrals (12) and (13) but not the y-momentum integral (14). The evolution of the two terms comprising the y-momentum integral (14)—yψ and y0(t)ψs, as well as their sum—is shown in Fig. 7, which does indicate the near-conservation of this integral of motion in all cases.

Analogous budgets for the energy conservation (12) are displayed in Fig. 8. The total energy of the regular flow Er, the term −(p² − a²)Er [which, here, only includes the first, time-dependent term in (12c)], and the term −Aψ|v(x) total| all have comparable magnitudes, with the sum of the three terms Er − (p² − a²)Er − Aψ|v(x) total| being much smaller than the individual terms.

A comparison of this sum with the time integrated dissipative term D1 + D2 shows that the energy changes are indeed due to the effect of dissipative terms, in line with Eq. (12a), which verifies a reasonable energy conservation in our finite-difference representation of (8).
Finally, Fig. 9 examines the enstrophy integral (13). In contrast to the energy integral, the frictional effects provide a somewhat more pronounced contribution to the enstrophy budget, which can be easily on the order of the contributions from both $S$ and $-\beta Ay_0$ terms in the large-friction, small-vortex simulations but exhibit a relative decrease in the smaller friction and/or larger-vortex simulations. The sum $S - \beta Ay_0$ monotonically decreases with time, as expected from (12c) and (13c), and is well approximated by the integrated sink $D_3$.

To summarize, our finite-difference version of (7)–(9) conserves all of the invariants (12)–(14) reasonably well, thus matching the corresponding conservation properties of the underlying continuous system.

**B. Cyclone trajectories**

At small times, within the linear regime, all of the solutions, irrespective of the value of $K$, follow the numerical estimates based on linearized Eq. (18), with vortex trajectories starting northward and then quickly bending toward west (Fig. 10, top row). Recall that the analytical solution (A6) and (A7) for the point-vortex case is identical to the numerical solution of the linearized problem, while it only matches the initial evolution of the linearized problem in other cases, due to neglecting the term $J(\psi + U_0y - V_0x, (p^2 - a^2)\psi_y)$ in (18) (this term exactly vanishes in the point-vortex case $p = a$) (see Sec. IV A and the Appendix). During the subsequent nonlinear inertial regime, the numerical trajectories deviate northward from the linear solution (which neglects self-interaction of the regular field). The duration of the inertial regime is dynamically limited by the frictional damping and, hence, decreases with increasing $K$ (see Sec. V C for the quantitative estimates of this duration).

The singular-vortex trajectories tend to become more zonally oriented in the large-vortex case $L_v = 2R_d$ [Fig. 10(c-1)] and, conversely, become more meridionally oriented in the small-vortex case.
$L_v = R_d/2$ [Fig. 10(a-1)]. The small-vortex solutions are characterized by a shorter inertial stage and are only able to reach relatively small westward velocities [Fig. 10(a-2)] in the course of the evolution compared to the point-vortex [Fig. 10(b-2)] and, especially, large-vortex solutions [Fig. 10(b-3)], whereas the magnitudes of the meridional velocity component seem to depend more on the superviscosity $K$ than on the size of the singular vortex, as per Figs. 10(a-3), 10(b-3), and 10(c-3).

To recap, the singular cyclone trajectories outside of the initial, linear stage of their development exhibit northwestward orientation; they tend to become more zonally oriented with smaller $K$ and larger $L_v$. Note that the latter two dependencies are in a sense consistent since the frictional damping $-K\nabla^2\psi$ in (8) is highly scale-selective, and hence, larger-scale anomalies are subject to smaller effective damping. The main long-term effect of friction seems to be in limiting the net zonal acceleration of the singular vortex and inducing an additional meridional drift of the vortex so that the frictional solutions never reach the long-time asymptotic behavior predicted by approximate inviscid solution (A6) and (A7), in which the singular vortex propagates westward with the limiting Rossby-wave velocity $\beta R_d^2$ (Reznik, 1990; 1992).

C. Frictional regime

An interesting and somewhat unexpected property of the singular-vortex evolution is the existence of a “postinertial” stage of evolution in which frictional dissipation plays an important role. This stage becomes most apparent in the solutions with large $K$ (and was not observed at all in the tiny-friction simulations).

1. Behavior of trajectories

The start of the frictional stage is marked by a kink in the singular-vortex trajectories (Fig. 10, top row) due to a relatively
sharp decrease in the westward zonal velocity of the singular vortex (Fig. 10, middle row) and speedup of its meridional propagation (Fig. 10, bottom row). For the large-friction point-vortex case [Fig. 10(b)], this transition to the frictional stage happens at $t \approx 15$ days, at which time the zonal component of the (westward) vortex velocity $U_0$ starts to decrease [Fig. 10(b-2)], and its (northward) meridional component $V_0$ starts to increase [Fig. 10(b-3)] until both become nearly steady at $t = 30$ days. The fully developed frictional stage is characterized by the uniform northwestward propagation of the singular vortex [Fig. 10(b-1)], with velocity components exhibiting damped oscillations around their final steady states [Figs. 10(b-2) and 10(b-3)]. Compared to the large-friction case, the medium-friction point-vortex solution [Fig. 10(b), green curves] starts the transition to the frictional regime a bit later at $t \approx 20$ days [based on the timing of the zonal-velocity time-derivative sign change; Fig. 10(b-2)] and reaches the quasistationary solution at $t \approx 60$ days, while the small-friction and tiny-friction solutions are unable to reach their respective frictional steady states during the course of the simulation (see further discussion of the friction-assisted steady states below).

2. Diagnostics using PV conservation

The onset of the frictional regime becomes particularly apparent in tracking the evolution of the regular flow’s potential vorticity $q$ at the center of the singular vortex (15). In the inviscid case, $q$ is constant, so the changes in $q$ in the course of the present numerical simulations reflect the frictional effects. In general, $q$ does remain constant or nearly constant during initial (linear and inertial) stages of the system’s evolution but eventually exhibits linear growth in the frictional regime (Fig. 11). This linear growth reflects the occurrence of the friction-assisted steady state, in which the regular streamfunction $\psi$ in the reference frame moving with the singular vortex
is approximately constant (Fig. 12), along with the singular vortex zonal and meridional velocities (Fig. 10); hence, $q = \beta y_0 = \text{const}$.

The onset of the frictional regime is a function of both the size of the vortex $L_v$ and superviscosity $K$ since even under constant $K$, the smaller-scale anomalies are damped more efficiently compared to larger-scale anomalies. These dependences are clearly evident in Fig. 11: the linear growth of $q$ starts later in the simulations with larger $L_v$ and/or smaller $K$. Based on the diagnosis of this linear growth, the friction-assisted regime was reached in all of the large-friction and medium-friction simulations, as well as in the small-friction small-vortex case; in all other cases, the simulations stopped before the onset of the frictional regime, but the precursors of this regime, such as the negative tendency in $q$ preceding the linear growth stage, are still evident in most of these cases. The only exception is the least-damped experiment with tiny $K$ and large $L_v = 2R_d$, in which $q$ is constant throughout the entire duration of the simulation (cf. Sec. IV).

Shown in Fig. 13 is the evolution of the potential vorticity $Q = \nabla^2 \psi - a^2 \psi + \beta y$ in the frictional regime of the medium-friction point-vortex simulation. As evident from the previous discussion, toward the end of the simulation, $Q$ exhibits a constant spatial structure in the reference frame centered at the singular vortex, with the background planetary vorticity $\beta y$ linearly increasing with time due to the uniform northwestward propagation of the singular vortex. Interestingly, this structure has a zonally oriented comma shape and a PV minimum at the center of the singular vortex. By contrast, the streamfunction maximum in the steady-state regime is shifted with respect to the center of the singular vortex (Fig. 12), resulting in the northwestward propagation of the vortex.
3. Considerations using integrals of motion

Some properties of the frictional stage can be predicted from conservation integrals (cf. Sec. IV A). For example, if the model (7)–(9) has an equilibrium (steady-state) solution, in which $\partial S/\partial t = 0$, then the enstrophy conservation (13a) gives

$$V_0 = -D_3/\beta A \geq 0,$$

where the equality sign only applies to the inviscid case $K = 0$. Thus, the equilibrium singular-cyclone motion in the system with nonzero frictional damping would have a positive northward component. Furthermore, in such a steady state, both $\psi_s$ and $\psi$ would remain constant in the reference frame associated with the singular vortex. Therefore, both components of the singular vortex velocity $U_0$, $V_0$ would also be constant, according to (9). In the same way, all of the terms comprising the energy integral [(12a)–(12f)] would be constant as well, and the time derivative on the left-hand side of (12a) would vanish. This means that in the equilibrium state,

$$D_2 = -D_1,$$

which imposes a frictional constraint on the long-time steady-state solution of the system (7)–(9), thus making it fundamentally different from the case with $K = 0$.

VI. SUMMARY AND DISCUSSION

In this paper, we developed a finite-difference numerical formulation to describe the evolution of discrete–continuous systems consisting of the singular vortices and regular flow generated by these vortices, in the context of a one-layer (or 1.5-layer) quasigeostrophic $\beta$-plane model. In this model, each Bessel singular vortex is replaced by the equivalent vortex of a finite amplitude (that is, without the singularity at the center of the vortex), consistent with the finite spatial resolution of the model. The model also contains scale-selective frictional damping, which only acts on the regular component of the flow. We verified our model by exploring, numerically, the evolution of an individual singular monopole, using parameter values typical for the middle-latitude atmosphere and a wide range of (super)viscosities. For small values of viscosity, the numerical model accurately reproduces the behavior expected from the theoretical considerations of inviscid singular monopoles (Reznik, 1990; 1992), as well as from numerous studies of distributed monopoles (e.g., Carnevale et al., 1991; Sutyrin and Flierl, 1994; Reznik and Dewar, 1994; Sutyrin et al., 1994; Reznik et al., 2000; and Lam and Dritshel, 2001). The model also conserves, approximately, several invariants of motion derived from the continuous equations (Reznik and Kizner, 2007a) and accurately represents their modifications in the presence of friction. Our numerical procedure can be readily generalized to the case of any number of singular vortices interacting with one another and with the regular flow. The numerical formulation and its substantiation constitute our first important result.

The evolution of a singular cyclone is shown to exhibit three main stages: (1) the linear stage, in which the regular Rossby-wave field generated by the cyclone is small; (2) the inertial stage, where self-interactions within the regular field become important; and (3) the frictional stage culminating in a friction-assisted steady state. The linear stage of the vortex evolution is characterized by the
development, in the near vortex regular flow field, of the secondary dipole circulation—the so called \( \beta \) gyres—which force the cyclone to move northwestern. We showed that at the end of the linear regime and throughout the inertial stage, the \( \beta \) gyres gradually disintegrate: the singular vortex merges with the anticyclonic lobe of the \( \beta \) gyres, while the cyclonic lobe separates from the rest of the system and dissipates. Thus, at large times, the singular cyclone in the Northern Hemisphere moves northwestern by forming a dipole-like system with the localized secondary regular-field anticyclone northeast of the singular-cyclone center. Revealing the existence of such a new mechanism of the \( \beta \) drift is our second important result.

Finally, our third important result is the discovery of the frictional regime, which follows the inertial stage of vortex evolution. This stage is observed for sufficiently large friction and is characterized by the cyclone’s uniform north-westward propagation and a regular field that has a constant spatial pattern in the reference frame associated with the singular vortex. Quasi-steady regimes were found earlier by Early et al. (2011) in a numerical study of quasi-inviscid distributed monopoles on the \( \beta \) plane. Their solutions probably correspond to our transitional quasisteady stage of the inertial regime, which is particularly long and pronounced in our large-vortex simulations with weak and tiny friction [Fig. 10(c)]; yet, these simulations never reach the friction-assisted regime.

The singular vortices embedded into an regular background flow have been studied extensively in the context of geophysical fluid dynamics (e.g., Gryanik and Tevs, 1989; Reznik, 1990; 1992; Wu and Emanuel, 1993; Kuhlbrodt and Nevis, 2000; Dunn et al., 2001; Gryanik et al., 2004; Reznik and Kizner, 2007a; 2007b; Southwick et al., 2015; Ryzhov and Koshel, 2016; Ryzhov and Sokolovskiy, 2016; Koshel et al., 2018; 2019; Reinaud et al., 2018; and Ryzhov et al., 2018) and plasma physics (cf. Kono et al., 1994; Kiwamoto et al., 2000; Schecter and Dubin, 2001; Driscoll et al., 2002; Dubin, 2003; Leoncini and Verga, 2013; and Tur and Yanovsky, 2017). We believe that the numerical formulation developed in the present work, despite a clearly idealized character of an underlying dynamical model, will help provide further insights into our understanding of complicated nonlinear vortex-flow systems, such as the tropical cyclones (cf. Wu and Emanuel, 1993). Considerations of the system of multiple singular vortices will be most useful when addressing many atmospheric problems, including eddy–mean-flow interaction (Kravtsov and Gulev, 2013), serial clustering of atmospheric cyclones (Mailler et al., 2006), and poleward deflection of midlatitude storm tracks (Tamarin and Kaspi, 2016), extratropical transition of tropical cyclones (Evans et al., 2017), among many others. The oceanic eddy field consisting of the many long-lived vortices of different scales and amplitudes (e.g., Early et al., 2011) can also be simulated in this way.

ACKNOWLEDGMENTS

This work was supported by the FASO State Assignment No. 0149-2019-0002 (development of theoretical framework in Sec. II), the Russian Foundation for Basic Research, Project No. 17-05-00094 (development of numerical formulation in Sec. III), and the Ministry of Education and Science of the Russian Federation, Grant No. 14.W03.31.0006 (numerical experiments in Secs. IV and V).

APPENDIX: \( \beta \) GYRES AND INITIAL TRAJECTORY OF THE SINGULAR VORTEX

In the reference frame associated with the singular vortex,

\[
x' = x - x_0(t), \quad y' = y - y_0(t),
\]

Eq. (8) becomes (omitting primes and setting \( K = 0 \)),

\[
\partial_t \Omega + J(\psi, \Omega) + \beta \partial_x \psi + (p - a^2)J(\psi + U_0 y - V_0 x, \psi) + \beta \partial_x \psi + J(\psi + U_0 y - V_0 x, \Omega) = 0.
\]

In (A2), the term \( \beta \partial_x \psi \) remains constant in time, while—in view of (9) and (10)—the regular streamfunction \( \psi \) and translation velocities \( U_0, V_0 \) are zero initially and grow in magnitude with time. This means that during some initial time interval, the sum \( \beta \partial_x \psi + J(\psi + U_0 y - V_0 x, \Omega) \) can be neglected compared to the first three terms in (A2); as a result, we obtain the approximate linear Eq. (18) describing the development of the \( \beta \) gyres. The period of time for which (18) remains approximately valid depends on the magnitude \( A \) of the singular vortex (7): the larger the value of \( A \), the longer this period (see the work of Reznik, 1992 for more in-depth discussion).

For the point vortex \( p = a \), the last term in (18) disappears and this equation takes a simpler form

\[
\partial_t (\Omega + \beta \psi) + J(\psi, \Omega + \beta \psi) = 0.
\]

For small and large vortices with \( p \neq a \), the last term can only be neglected [and, hence, (A3) is only valid] at the very initial stage of the system’s evolution, until \( \psi \approx \beta p^{-1} U_0, V_0 \approx \beta p^{-2} \) (Reznik, 1992).

The approximate problem (A3), (9), and (10) can be easily solved. In polar coordinates \((x = r \cos \theta, y = r \sin \theta)\), we have

\[
\nabla^2 \psi - a^2 \psi = \beta r (\cos bt - 1) \sin \theta - \beta r \sin bt \cos \theta,
\]

with

\[
b = b(r) = \frac{ApK_1(pr)}{2\pi r}.
\]

The solution of (A4) and (A5) is

\[
\psi = \varphi_1(r, t) \sin \theta + \varphi_2(r, t) \cos \theta,
\]

\[
\varphi_1(r, t) = \beta I_1(r) \int_0^\infty \tau^2 K_1(\tau r)(1 - \cos bt) d\tau + \beta K_1(r) \int_0^\infty \tau^2 I_1(\tau r)(1 - \cos bt) d\tau,
\]

\[
\varphi_2(r, t) = \beta I_1(r) \int_0^\infty \tau^2 K_1(\tau r) \sin bt d\tau + \beta K_1(r) \int_0^\infty \tau^2 I_1(\tau r) \sin bt d\tau,
\]

which is a generalization of formula (27) in the work of Reznik (1992) (who considered the case \( p \approx a = R_2^{-1} \)) for the singular vortex of an arbitrary size. The singular vortex velocities (and then trajectories) can be found from (9) and (A6),

\[
U_0 = \dot{x}_0 = -\frac{1}{2} \beta R_2^{-1} \int_0^\infty \tau^2 K_1(\tau r)(1 - \cos bt) d\tau,
\]

\[
V_0 = \dot{y}_0 = -\beta R_2^{-1} \int_0^\infty \tau^2 I_1(\tau r)(1 - \cos bt) d\tau.
\]
\[ V_0 = j_0 = \frac{1}{2} \beta R^3 \int_{-\infty}^{\infty} r^2 K_i(\alpha r) \sin bt \, dr \]  

(A7b)

[cf. Reznik, 1992, Eq. (28)].

REFERENCES


